Relationship to exponential function and complex numbers

Euler's formula illustrated with the three dimensional helix, starting with the 2-D orthogonal components of the unit circle, sine and cosine (using $\theta = t$).
It can be shown from the series definitions[14] that the sine and cosine functions are the imaginary and real parts, respectively, of the complex exponential function when its argument is purely imaginary:

This identity is called Euler's formula. In this way, trigonometric functions become essential in the geometric interpretation of complex analysis. For example, with the above identity, if one considers the unit circle in the complex plane, parametrized by e^{ix}, and as above, we can parametrize this circle in terms of cosines and sines, the relationship between the complex exponential and the trigonometric functions becomes more apparent.
Euler's formula can also be used to derive some trigonometric identities, by writing sine and cosine as:

Furthermore, this allows for the definition of the trigonometric functions for complex arguments z:

where $i^2 = -1$. The sine and cosine defined by this are entire functions. Also, for purely real x,

It is also sometimes useful to express the complex sine and cosine functions in terms of the real and imaginary parts of their arguments.

This exhibits a deep relationship between the complex sine and cosine functions and their real (*sin*, *cos*) and hyperbolic real (*sinh*, *cosh*) counterparts.

Complex graphs

In the following graphs, the domain is the complex plane pictured, and the range values are indicated at each point by color. Brightness indicates the size (absolute value) of the range value, with black being zero. Hue varies with argument, or angle, measured from the positive real axis. (more)

Trigonometric functions in the complex plane

Definitions via differential equations

Both the sine and cosine functions satisfy the differential equation:

That is to say, each is the additive inverse of its own second derivative. Within the 2-dimensional function space *V* consisting of all solutions of this equation,

- the sine function is the unique solution satisfying the initial condition and
- the cosine function is the unique solution satisfying the initial condition .

Since the sine and cosine functions are linearly independent, together they form a basis of *V*. This method of defining the sine and cosine functions is essentially equivalent to using Euler's formula. (See linear differential equation.) It turns out that this differential equation can be used not only to define the sine and cosine functions but also to prove the trigonometric identities for the sine and cosine functions.

Further, the observation that sine and cosine satisfies $y'' = -y$ means that they are eigenfunctions of the second-derivative operator.

The tangent function is the unique solution of the nonlinear differential equation

satisfying the initial condition $y(0) = 0$. There is a very interesting visual proof that the tangent function satisfies this differential equation.[15]

The significance of radians

Radians specify an angle by measuring the length around the path of the unit circle and constitute a special argument to the sine and cosine functions. In particular, only sines and cosines that map radians to ratios satisfy the differential equations that classically describe them. If an argument to sine or cosine in radians is scaled by frequency,

then the derivatives will scale by *amplitude*.

Here, *k* is a constant that represents a mapping between units. If *x* is in degrees, then

This means that the second derivative of a sine in degrees does not satisfy the differential equation

but rather

The cosine's second derivative behaves similarly.
This means that these sines and cosines are different functions, and that the fourth derivative of sine will be sine again only if the argument is in radians.

Identities

Main article: List of trigonometric identities
Many identities interrelate the trigonometric functions. Among the most frequently used is the Pythagorean identity, which states that for any angle, the square of the sine plus the square of the cosine is 1. This is easy to see by studying a right triangle of hypotenuse 1 and applying the Pythagorean theorem. In symbolic form, the Pythagorean identity is written

where is standard notation for
Other key relationships are the sum and difference formulas, which give the sine and cosine of the sum and difference of two angles in terms of sines and cosines of the angles themselves. These can be derived geometrically, using arguments that date to Ptolemy. One can also produce them algebraically using Euler's formula.
Sum

Subtraction

These in turn lead to the following three-angle formulae:

When the two angles are equal, the sum formulas reduce to simpler equations known as the double-angle formulae.

When three angles are equal, the three-angle formulae simplify to

These identities can also be used to derive the product-to-sum identities that were used in antiquity to transform the product of two numbers into a sum of numbers and greatly speed operations, much like the logarithm function.

Calculus

For integrals and derivatives of trigonometric functions, see the relevant sections of Differentiation of trigonometric functions, Lists of integrals and List of integrals of trigonometric functions. Below is the list of the derivatives and integrals of the six basic trigonometric functions. The number *C* is a constant of integration.

Definitions using functional equations

In mathematical analysis, one can define the trigonometric functions using functional equations based on properties like the difference formula. Taking as given these formulas, one can prove that only two real functions satisfy those conditions. Symbolically, we say that there exists exactly one pair of real functions — and — such that for all real numbers and , the following equation hold:[16]

with the added condition that

Other derivations, starting from other functional equations, are also possible, and such derivations can be extended to the complex numbers. As an example, this derivation can be used to define trigonometry in Galois fields.

Computation

The computation of trigonometric functions is a complicated subject, which can today be avoided by most people because of the widespread availability of computers and scientific calculators that provide built-in trigonometric functions for any angle. This section, however, describes details of their computation in three important contexts: the historical use of trigonometric tables, the modern techniques used by computers, and a few "important" angles where simple exact values are easily found.

The first step in computing any trigonometric function is range reduction—reducing the given angle to a "reduced angle" inside a small range of angles, say 0 to $\pi/2$, using the periodicity and symmetries of the trigonometric functions.

Main article: Generating trigonometric tables

Prior to computers, people typically evaluated trigonometric functions by interpolating from a detailed table of their values, calculated to many significant figures. Such tables have been available for as long as trigonometric functions have been described (see History below), and were typically generated by repeated application of the half-angle and angle-addition identities starting from a known value (such as $\sin(\pi/2) = 1$).

Modern computers use a variety of techniques.[17] One common method, especially on higher-end processors with floating point units, is to combine a polynomial or rational approximation (such as Chebyshev approximation, best uniform approximation, and Padé approximation, and typically for higher or variable precisions, Taylor and Laurent series) with range reduction and a table lookup—they first look up the closest angle in a small table, and then use the polynomial to compute the correction.[18] Devices that lack hardware multipliers often use an algorithm called CORDIC (as well as related techniques), which uses only addition, subtraction, bitshift, and table lookup. These methods are commonly implemented in hardware floating-point units for performance reasons.

For very high precision calculations, when series expansion convergence becomes too slow, trigonometric functions can be approximated by the arithmetic-geometric mean, which itself approximates the trigonometric function by the (complex) elliptic integral.[19]

Main article: Exact trigonometric constants

Finally, for some simple angles, the values can be easily computed by hand using the Pythagorean theorem, as in the following examples. For example, the sine, cosine and tangent of any integer multiple of radians (3°) can be found exactly by hand.

Consider a right triangle where the two other angles are equal, and therefore are both radians (45°). Then the length of side *b* and the length of side *a* are equal; we can choose . The values of sine, cosine and tangent of an angle of radians (45°) can then be found using the Pythagorean theorem:

Therefore:

Computing trigonometric functions from an equilateral triangle

To determine the trigonometric functions for angles of π/3 radians (60 degrees) and π/6 radians (30 degrees), we start with an equilateral triangle of side length 1. All its angles are π/3 radians (60 degrees). By dividing it into two, we obtain a right triangle with π/6 radians (30 degrees) and π/3 radians (60 degrees) angles. For this triangle, the shortest side = 1/2, the next largest side =(√3)/2 and the hypotenuse = 1. This yields:

Special values in trigonometric functions

There are some commonly used special values in trigonometric functions, as shown in the following table.

Function							
sin							
cos							
tan							
cot	[20]						
sec							
csc	[20]						

The symbol here represents the point at infinity on the real projective line, the limit on the extended real line is on one side and on the other.

Inverse functions

Main article: Inverse trigonometric functions

The trigonometric functions are periodic, and hence not injective, so strictly they do not have an inverse function. Therefore, to define an inverse function we must restrict their domains so that the trigonometric function is bijective. In the following, the functions on the left are *defined* by the equation on the right; these are not proved identities. The principal inverses are usually defined as:

Function	Definition	Value Field

The notations $\sin^{-1}$ and $\cos^{-1}$ are often used for arcsin and arccos, etc. When this notation is used, the inverse functions could be confused with the multiplicative inverses of the functions. The notation using the "arc-" prefix avoids such confusion, though "arcsec" can be confused with "arcsecond".

Just like the sine and cosine, the inverse trigonometric functions can also be defined in terms of infinite series. For example, These functions may also be defined by proving that they are antiderivatives of other functions. The arcsine, for example, can be written as the following integral:

Analogous formulas for the other functions can be found at Inverse trigonometric functions. Using the complex logarithm, one can generalize all these functions to complex arguments:

Connection to the inner product

In an inner product space, the angle between two non-zero vectors is defined to be

Properties and applications

Main article: Uses of trigonometry

The trigonometric functions, as the name suggests, are of crucial importance in trigonometry, mainly because of the following two results.

Law of sines

The law of sines states that for an arbitrary triangle with sides a, b, and c and angles opposite those sides A, B and C:

where is the area of the triangle, or, equivalently,

where R is the triangle's circumradius.

A Lissajous curve, a figure formed with a trigonometry-based function.

It can be proven by dividing the triangle into two right ones and using the above definition of sine. The law of sines is useful for computing the lengths of the unknown sides in a triangle if two angles and one side are known. This is a common situation occurring in *triangulation*, a technique to determine unknown distances by measuring two angles and an accessible enclosed distance.

Law of cosines

The law of cosines (also known as the cosine formula or cosine rule) is an extension of the Pythagorean theorem:

or equivalently,

In this formula the angle at *C* is opposite to the side *c*. This theorem can be proven by dividing the triangle into two right ones and using the Pythagorean theorem.

The law of cosines can be used to determine a side of a triangle if two sides and the angle between them are known. It can also be used to find the cosines of an angle (and consequently the angles themselves) if the lengths of all the sides are known.

Law of tangents

Main article: Law of tangents

The following all form the law of tangents[21]

The explanation of the formulae in words would be cumbersome, but the patterns of sums and differences; for the lengths and corresponding opposite angles, are apparent in the theorem.

Law of cotangents

Main article: Law of cotangents

If

(the radius of the inscribed circle for the triangle) and

(the semi-perimeter for the triangle), then the following all form the law of cotangents[22]

It follows that

In words the theorem is: the cotangent of a half-angle equals the ratio of the semi-perimeter minus the opposite side to the said angle, to the inradius for the triangle.

Periodic functions

An animation of the additive synthesis of a square wave with an increasing number of harmonics

Sinusoidal basis functions (bottom) can form a sawtooth wave (top) when added. All the basis functions have nodes at the nodes of the sawtooth, and all but the fundamental ($k = 1$) have additional nodes. The oscillation seen about the sawtooth when k is large is called the Gibbs phenomenon

The trigonometric functions are also important in physics. The sine and the cosine functions, for example, are used to describe simple harmonic motion, which models many natural phenomena, such as the movement of a mass attached to a spring and, for small angles, the pendular motion of a mass hanging by a string. The sine and cosine functions are one-dimensional projections of uniform circular motion.

Trigonometric functions also prove to be useful in the study of general periodic functions. The characteristic wave patterns of periodic functions are useful for modeling recurring phenomena such as sound or light waves.[23]

Under rather general conditions, a periodic function $f(x)$ can be expressed as a sum of sine waves or cosine waves in a Fourier series.[24] Denoting the sine or cosine basis functions by ϕ_k, the expansion of the periodic function $f(t)$ takes the form:

For example, the square wave can be written as the Fourier series

In the animation of a square wave at top right it can be seen that just a few terms already produce a fairly good approximation. The superposition of several terms in the expansion of a sawtooth wave are shown underneath.

History

Main article: History of trigonometric functions

While the early study of trigonometry can be traced to antiquity, the trigonometric functions as they are in use today were developed in the medieval period. The chord function was discovered by Hipparchus of Nicaea (180–125 BC) and Ptolemy of Roman Egypt (90–165 AD).

The functions sine and cosine can be traced to the *jyā* and *koti-jyā* functions used in Gupta period Indian astronomy (*Aryabhatiya*, *Surya Siddhanta*), via translation from Sanskrit to Arabic and then from Arabic to Latin.[25]

All six trigonometric functions in current use were known in Islamic mathematics by the 9th century, as was the law of sines, used in solving triangles.[26] al-Khwārizmī produced tables of sines, cosines and tangents. They were studied by authors including Omar Khayyám, Bhāskara II, Nasir al-Din al-Tusi, Jamshīd al-Kāshī (14th century), Ulugh Beg (14th century), Regiomontanus (1464), Rheticus, and Rheticus' student Valentinus Otho.[*citation needed*]

Madhava of Sangamagrama (c. 1400) made early strides in the analysis of trigonometric functions in terms of infinite series.[27]

The first published use of the abbreviations 'sin', 'cos', and 'tan' is by the 16th century French mathematician Albert Girard.

In a paper published in 1682, Leibniz proved that sin x is not an algebraic function of x.[28]

Leonhard Euler's *Introductio in analysin infinitorum* (1748) was mostly responsible for establishing the analytic treatment of trigonometric functions in Europe, also defining them as infinite series and presenting "Euler's formula", as well as the near-modern abbreviations *sin., cos., tang., cot., sec.,* and *cosec*.[6]

A few functions were common historically, but are now seldom used, such as the chord ($\mathrm{crd}(\theta) = 2 \sin(\theta/2)$), the versine ($\mathrm{versin}(\theta) = 1 - \cos(\theta) = 2 \sin^2(\theta/2)$) (which appeared in the earliest tables[6]), the haversine ($\mathrm{haversin}(\theta) = \mathrm{versin}(\theta) / 2 = \sin^2(\theta/2)$), the exsecant ($\mathrm{exsec}(\theta) = \sec(\theta) - 1$) and the excosecant ($\mathrm{excsc}(\theta) = \mathrm{exsec}(\pi/2 - \theta) = \csc(\theta) - 1$). Many more relations between these functions are listed in the article about trigonometric identities.

Etymologically, the word *sine* derives from the Sanskrit word for half the chord, *jya-ardha*, abbreviated to *jiva*. This was transliterated in Arabic as *jiba*, written *jb*, vowels not being written in Arabic. Next, this transliteration was mis-translated in the 12th century into Latin as *sinus*, under the mistaken impression that *jb* stood for the word *jaib*, which means "bosom" or "bay" or "fold" in Arabic, as does *sinus* in Latin.[29] Finally, English usage converted the Latin word *sinus* to *sine*.[30] The word *tangent* comes from Latin *tangens* meaning "touching", since the line *touches* the circle of unit radius, whereas *secant* stems from Latin *secans* — "cutting" — since the line *cuts* the circle.

See also

- All Students Take Calculus — a mnemonic for recalling the signs of trigonometric functions in a particular quadrant of a Cartesian plane
- Aryabhata's sine table
- Bhaskara I's sine approximation formula
- Euler's formula
- Gauss's continued fraction — a continued fraction definition for the tangent function
- Generalized trigonometry
- Generating trigonometric tables
- Hyperbolic function
- List of periodic functions
- List of trigonometric identities
- Madhava series
- Madhava's sine table
- Polar sine — a generalization to vertex angles
- Proofs of trigonometric identities
- Table of Newtonian series
- Unit vector (explains direction cosines)

Notes

1 ^ Oxford English Dictionary, sine, *n.*[2]
2 ^ Oxford English Dictionary, cosine, *n*.
3 ^ Oxford English Dictionary, tangent, *adj. and n*.
4 ^ Oxford English Dictionary, secant, *adj*. and *n*.
5 ^ Heng, Cheng and Talbert, "Additional Mathematics", page 228
6 ^ *a b c* See Boyer (1991).
7 ^ See Maor (1998)
8 ^ See Ahlfors, pages 43–44.
9 ^ Abramowitz; Weisstein.
10 ^ Stanley, Enumerative Combinatorics, Vol I., page 149
11 ^ Stanley, Enumerative Combinatorics, Vol I
12 ^ *Aigner, Martin; Ziegler, Günter M. (2000). Proofs from THE BOOK*

(Second ed.). Springer-Verlag. p. 149. ISBN 978-3-642-00855-9.
13 ^ *Remmert, Reinhold (1991). Theory of complex functions. Springer. p. 327. ISBN 0-387-97195-5.*, Extract of page 327
14 ^ For a demonstration, see Euler's formula#Using power series
15 ^ *Needham, Tristan. Visual Complex Analysis. ISBN 0-19-853446-9.*
16 ^ *Kannappan, Palaniappan (2009). Functional Equations and Inequalities with Applications. Springer. ISBN 978-0387894911.*
17 ^ Kantabutra.
18 ^ However, doing that while maintaining precision is nontrivial, and methods like Gal's accurate tables, Cody and Waite reduction, and Payne and Hanek reduction algorithms can be used.
19 ^ *"R. P. Brent, "Fast Multiple-Precision Evaluation of Elementary Functions", J. ACM 23, 242 (1976).".*
20 ^ *a b c d* Abramowitz, Milton and Irene A. Stegun, p.74
21 ^ The Universal Encyclopaedia of Mathematics, Pan Reference Books, 1976, page 529. English version George Allen and Unwin, 1964. Translated from the German version Meyers Rechenduden, 1960.
22 ^ The Universal Encyclopaedia of Mathematics, Pan Reference Books, 1976, page 530. English version George Allen and Unwin, 1964. Translated from the German version Meyers Rechenduden, 1960.
23 ^ *Stanley J Farlow (1993). Partial differential equations for scientists and engineers (Reprint of Wiley 1982 ed.). Courier Dover Publications. p. 82. ISBN 0-486-67620-X.*
24 ^ See for example, *Gerald B Folland (2009). "Convergence and completeness". Fourier Analysis and its Applications (Reprint of Wadsworth & Brooks/Cole 1992 ed.). American Mathematical Society. pp. 77 ff. ISBN 0-8218-4790-2.*
25 ^ Boyer, Carl B. (1991). A History of Mathematics (Second ed.). John Wiley & Sons, Inc.. ISBN 0-471-54397-7, p. 210.
26 ^ *Owen Gingerich (1986). "Islamic Astronomy" 254. Scientific American. p. 74. Archived from the original on 2013-10-19. Retrieved 2010-07-13.*
27 ^ *J J O'Connor and E F Robertson. "Madhava of Sangamagrama". School of Mathematics and Statistics University of St Andrews, Scotland. Retrieved 2007-09-08.*
28 ^ *Nicolás Bourbaki (1994). Elements of the History of Mathematics. Springer.*
29 ^ See Maor (1998), chapter 3, regarding the etymology.

30 ^ "*Clark University*".

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External links

Wikibooks has a book on the topic of: *Trigonometry*

- *Hazewinkel, Michiel, ed. (2001), "Trigonometric functions", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4*
- Visionlearning Module on Wave Mathematics

- GonioLab[dead link] Visualization of the unit circle, trigonometric and hyperbolic functions

NewPP limit reportCPU time usage: 4.249 secondsReal time usage: 73.997 secondsPreprocessor visited node count: 4066/1000000Preprocessor generated node count: 13443/1000000Post‐expand include size: 41148/2097152 bytesTemplate argument size: 2867/2097152 bytesHighest expansion depth: 11/40Expensive parser function count: 5/100Lua time usage: 0.206/10.000 secondsLua memory usage: 2.07 MB/953.67 MB Transclusion expansion time report (%,ms,calls,template)100.00% 46528.404 1 - -total 18.88% 8784.765 1 - Template:Reflist 10.55% 4908.242 5 - Template:Fix 9.22% 4289.578 3 - Template:Citation_needed 8.02% 3731.128 10 - Template:Category_handler 4.09% 1902.142 7 - Template:Cite_book 4.01% 1868.040 1 - Template:Trigonometry 3.86% 1793.807 2 - Template:Dead_link 3.74% 1739.608 5 - Template:Fix/category 3.39% 1575.678 1 - Template:Sidebar Saved in parser cache with key my_wiki:pcache:idhash:30367-1!*!0!!en!5!*!math=0 and timestamp 20151019160246 and revision id 682520343