Quadrants and octants

Main articles: Octant (solid geometry) and Quadrant (plane geometry)

The four quadrants of a Cartesian coordinate system.

The axes of a two-dimensional Cartesian system divide the plane into four infinite regions, called quadrants, each bounded by two half-axes. These are often numbered from 1st to 4th and denoted by Roman numerals: I (where the signs of the two coordinates are +,+), II (−,+), III (−,−), and IV (+,−). When the axes are drawn according to the mathematical custom, the numbering goes counter-clockwise starting from the upper right ("north-east") quadrant.

Similarly, a three-dimensional Cartesian system defines a division of space into eight regions or octants, according to the signs of the coordinates of the points. The convention used for naming a specific octant is to list its signs, e.g. (+ + +) or (− + −). The generalization of the quadrant and octant to an arbitrary number of dimensions is the orthant, and a similar naming system applies.

Cartesian formulae for the plane

Distance between two points

The Euclidean distance between two points of the plane with Cartesian coordinates and is

This is the Cartesian version of Pythagoras's theorem. In three-dimensional space, the distance between points and is

which can be obtained by two consecutive applications of Pythagoras' theorem.

Euclidean transformations

The Euclidean transformations or Euclidean motions are the (bijective) mappings of points of the Euclidean plane to themselves which preserve distances between points. There are four types of these mappings (also called isometries): translations, rotations, reflections and glide reflections.[5]

Translation

Translating a set of points of the plane, preserving the distances and directions between them, is equivalent to adding a fixed pair of numbers (a, b) to the Cartesian coordinates of every point in the set. That is, if the original coordinates of a point are (x, y), after the translation they will be

Rotation

To rotate a figure counterclockwise around the origin by some angle is equivalent to replacing every point with coordinates (x,y) by the point with coordinates (x',y'), where

Thus:

Reflection

If (x, y) are the Cartesian coordinates of a point, then ($-x$, y) are the coordinates of its reflection across the second coordinate axis (the Y-axis), as if that line were a mirror. Likewise, (x, $-y$) are the coordinates of its reflection across the first coordinate axis (the X-axis). In more generality, reflection across a line through the origin making an angle with the x-axis, is equivalent to replacing every point with coordinates (x, y) by the point with coordinates (x',y'), where

Thus:

Glide reflection

A glide reflection is the composition of a reflection across a line followed by a translation in the direction of that line. It can be seen that the order of these operations does not matter (the translation can come first, followed by the reflection).

General matrix form of the transformations

These Euclidean transformations of the plane can all be described in a uniform way by using matrices. The result of applying a Euclidean transformation to a point is given by the formula

where A is a 2×2 orthogonal matrix and $b = (b_1, b_2)$ is an arbitrary ordered pair of numbers;[6] that is, where

[Note the use of row vectors for point coordinates and that the matrix is written on the right.]

To be *orthogonal*, the matrix A must have orthogonal rows with same Euclidean length of one, that is,

and

This is equivalent to saying that A times its transpose must be the identity matrix. If these conditions do not hold, the formula describes a more general affine transformation of the plane provided that the determinant of A is not zero.

The formula defines a translation if and only if A is the identity matrix. The transformation is a rotation around some point if and only if A is a rotation matrix, meaning that

A reflection or glide reflection is obtained when,

Assuming that translation is not used transformations can be combined by simply multiplying the associated transformation matrices.

Affine transformation

Another way to represent coordinate transformations in Cartesian coordinates is through affine transformations. In affine transformations an extra dimension is added and all points are given a value of 1 for this extra dimension. The advantage of doing this is that point translations can be specified in the final column of matrix A. In this way, all of the euclidean transformations become transactable as matrix point multiplications. The affine transformation is given by:

[Note the matrix A from above was transposed. The matrix is on the left and column vectors for point coordinates are used.]

Using affine transformations multiple different euclidean transformations including translation can be combined by simply multiplying the corresponding matrices.

Scaling

An example of an affine transformation which is not a Euclidean motion is given by scaling. To make a figure larger or smaller is equivalent to multiplying the Cartesian coordinates of every point by the same positive number m. If (x, y) are the coordinates of a point on the original figure, the corresponding point on the scaled figure has coordinates

If m is greater than 1, the figure becomes larger; if m is between 0 and 1, it becomes smaller.

Shearing

A shearing transformation will push the top of a square sideways to form a parallelogram. Horizontal shearing is defined by:

Shearing can also be applied vertically:

Orientation and handedness

Main article: Orientation (mathematics)

See also: right-hand rule and Axes conventions

In two dimensions

The right hand rule.

Fixing or choosing the x-axis determines the y-axis up to direction. Namely, the y-axis is necessarily the perpendicular to the x-axis through the point marked 0 on the x-axis. But there is a choice of which of the two half lines on the perpendicular to designate as positive and which as negative. Each of these two choices determines a different orientation (also called *handedness*) of the Cartesian plane.

The usual way of orienting the axes, with the positive x-axis pointing right and the positive y-axis pointing up (and the x-axis being the "first" and the y-axis the "second" axis) is considered the *positive* or *standard* orientation, also called the *right-handed* orientation.

A commonly used mnemonic for defining the positive orientation is the *right hand rule*. Placing a somewhat closed right hand on the plane with the thumb pointing up, the fingers point from the x-axis to the y-axis, in a positively oriented coordinate system.

The other way of orienting the axes is following the *left hand rule*, placing the left hand on the plane with the thumb pointing up.

3D Cartesian Coordinate Handedness

When pointing the thumb away from the origin along an axis towards positive, the curvature of the fingers indicates a positive rotation along that axis.

Regardless of the rule used to orient the axes, rotating the coordinate system will preserve the orientation. Switching any two axes will reverse the orientation, but switching both will leave the orientation unchanged.

In three dimensions

Fig. 7 – The left-handed orientation is shown on the left, and the right-handed on the right.

Fig. 8 – The right-handed Cartesian coordinate system indicating the coordinate planes.

Once the *x*- and *y*-axes are specified, they determine the line along which the *z*-axis should lie, but there are two possible directions on this line. The two possible coordinate systems which result are called 'right-handed' and 'left-handed'. The standard orientation, where the *xy*-plane is horizontal and the *z*-axis points up (and the *x*- and the *y*-axis form a positively oriented two-dimensional coordinate system in the *xy*-plane if observed from *above* the *xy*-plane) is called right-handed or positive. The name derives from the right-hand rule. If the index finger of the right hand is pointed forward, the middle finger bent inward at a right angle to it, and the thumb placed at a right angle to both, the three fingers indicate the relative directions of the *x*-, *y*-, and *z*-axes in a *right-handed* system. The thumb indicates the *x*-axis, the index finger the *y*-axis and the middle finger the *z*-axis. Conversely, if the same is done with the left hand, a left-handed system results.

Figure 7 depicts a left and a right-handed coordinate system. Because a three-dimensional object is represented on the two-dimensional screen, distortion and ambiguity result. The axis pointing downward (and to

the right) is also meant to point *towards* the observer, whereas the "middle" axis is meant to point *away* from the observer. The red circle is *parallel* to the horizontal *xy*-plane and indicates rotation from the *x*-axis to the *y*-axis (in both cases). Hence the red arrow passes *in front of* the *z*-axis.
Figure 8 is another attempt at depicting a right-handed coordinate system. Again, there is an ambiguity caused by projecting the three-dimensional coordinate system into the plane. Many observers see Figure 8 as "flipping in and out" between a convex cube and a concave "corner". This corresponds to the two possible orientations of the coordinate system. Seeing the figure as convex gives a left-handed coordinate system. Thus the "correct" way to view Figure 8 is to imagine the *x*-axis as pointing *towards* the observer and thus seeing a concave corner.

Representing a vector in the standard basis

A point in space in a Cartesian coordinate system may also be represented by a position vector, which can be thought of as an arrow pointing from the origin of the coordinate system to the point.[7] If the coordinates represent spatial positions (displacements), it is common to represent the vector from the origin to the point of interest as . In two dimensions, the vector from the origin to the point with Cartesian coordinates (x, y) can be written as:

where , and are unit vectors in the direction of the *x*-axis and *y*-axis respectively, generally referred to as the *standard basis* (in some application areas these may also be referred to as versors). Similarly, in three dimensions, the vector from the origin to the point with Cartesian coordinates can be written as:[8]

where is the unit vector in the direction of the z-axis.
There is no *natural* interpretation of multiplying vectors to obtain another vector that works in all dimensions, however there is a way to use complex numbers to provide such a multiplication. In a two dimensional cartesian plane, identify the point with coordinates (x, y) with the complex number $z = x + iy$. Here, i is the imaginary unit and is

identified with the point with coordinates (0, 1), so it is not the unit vector in the direction of the *x*-axis. Since the complex numbers can be multiplied giving another complex number, this identification provides a means to "multiply" vectors. In a three dimensional cartesian space a similar identification can be made with a subset of the quaternions.

Applications

Cartesian coordinates are an abstraction that have a multitude of possible applications in the real world. However, three constructive steps are involved in superimposing coordinates on a problem application. 1) Units of distance must be decided defining the spatial size represented by the numbers used as coordinates. 2) An origin must be assigned to a specific spatial location or landmark, and 3) the orientation of the axes must be defined using available directional cues for (n-1) of the n axes.

Consider as an example superimposing 3D Cartesian coordinates over all points on the Earth (i.e. geospatial 3D). What units make sense? Kilometers are a good choice, since the original definition of the kilometer was geospatial...10,000 km equalling the surface distance from the Equator to the North Pole. Where to place the origin? Based on symmetry, the gravitational center of the Earth suggests a natural landmark (which can be sensed via satellite orbits). Finally, how to orient X, Y and Z axis directions? The axis of Earth's spin provides a natural direction strongly associated with "up vs. down", so positive Z can adopt the direction from geocenter to North Pole. A location on the Equator is needed to define the X-axis, and the Prime Meridian stands out as a reference direction, so the X-axis takes the direction from geocenter out to [0 degrees longitude, 0 degrees latitude]. Note that with 3 dimensions, and two perpendicular axes directions pinned down for X and Z, the Y-axis is determined by the first two choices. In order to obey the right hand rule, the Y-axis must point out from the geocenter to [90 degrees longitude, 0 degrees latitude]. So what are the geocentric coordinates of the Empire State Building in New York City? Using [longitude = −73.985656, latitude = 40.748433], Earth radius = 40,000/2π, and transforming from spherical --> Cartesian coordinates, you can estimate the geocentric coordinates of the Empire State Building, [x, y, z] = [1330.53 km, –4635.75 km, 4155.46 km].

GPS navigation relies on such geocentric coordinates.
In engineering projects, agreement on the definition of coordinates is a crucial foundation. One cannot assume that coordinates come predefined for a novel application, so knowledge of how to erect a coordinate system where there is none is essential to applying René Descartes' ingenious thinking.
While spatial apps employ identical units along all axes, in business and scientific apps, each axis may have different units of measurement associated with it (such as kilograms, seconds, pounds, etc.). Although four- and higher-dimensional spaces are difficult to visualize, the algebra of Cartesian coordinates can be extended relatively easily to four or more variables, so that certain calculations involving many variables can be done. (This sort of algebraic extension is what is used to define the geometry of higher-dimensional spaces.) Conversely, it is often helpful to use the geometry of Cartesian coordinates in two or three dimensions to visualize algebraic relationships between two or three of many non-spatial variables.
The graph of a function or relation is the set of all points satisfying that function or relation. For a function of one variable, f, the set of all points (x, y), where $y = f(x)$ is the graph of the function f. For a function g of two variables, the set of all points (x, y, z), where $z = g(x, y)$ is the graph of the function g. A sketch of the graph of such a function or relation would consist of all the salient parts of the function or relation which would include its relative extrema, its concavity and points of inflection, any points of discontinuity and its end behavior. All of these terms are more fully defined in calculus. Such graphs are useful in calculus to understand the nature and behavior of a function or relation.

See also

- Horizontal and vertical
- Jones diagram, which plots four variables rather than two.
- Orthogonal coordinates
- Polar coordinate system
- Spherical coordinate system

Notes

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External links

- Cartesian Coordinate System
- Printable Cartesian Coordinates
- Cartesian coordinates at PlanetMath.org.
- MathWorld description of Cartesian coordinates
- Coordinate Converter – converts between polar, Cartesian and spherical coordinates
- Coordinates of a point Interactive tool to explore coordinates of a point
- open source JavaScript class for 2D/3D Cartesian coordinate system manipulation

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